Distribution of escape times in a driven stochastic model

J. M. Casado and M. Morillo
Área de Física Teórica, Universidad de Sevilla, Apartado Correos 1065, Sector Sur. 41080 Sevilla, Spain
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We analyze the probability distribution of escape times out of a metastable well for a stochastic model driven by a large-amplitude sinusoidal time-dependent field. The system obeys a Langevin equation which is solved numerically by generating stochastic trajectories, both for a white and for an Ornstein-Uhlenbeck noise. The probability distribution changes from monomodal to multimodal as the noise strength is increased. The average escape time shows a nonmonotonic behavior with the noise intensity which is associated with the change in the structure of the probability distribution. For a noise with a correlation time much longer than the period of the driving field, the resonance effects are enhanced with respect to the white-noise case.

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I. INTRODUCTION

The interplay of noise and nonlinearity in systems externally driven by time-dependent forces has been the subject of recent interest, both theoretically and experimentally. In particular, when the external force is time periodic, there are a number of resonant effects in a variety of physical problems and which have been included under the terminology of stochastic resonance (SR) [1]. Most of the theoretical analysis has been carried out for bistable models in which the relevant degree of freedom has two attractors in the absence of noise and driving terms.

In a recent work, Dayan, Gitterman, and Weiss [2] have explored the escape of a particle from a single potential well under the combined influence of white noise and a sinusoidal time-dependent external force. In the deterministic limit, the dynamics of the system is described by the equation

$$\frac{dx(t)}{dt} = -U'(x) + S\cos(\Omega t)\ , \tag{1}$$

where $S\cos\Omega t$ represents the external force. The unbounded potential $U(x) = -x^2/2 + x^3/3$ has two stationary points at $x=0$ and 1 corresponding, respectively, to a local maximum and minimum. Weiss and co-workers discuss the dependence of the escape time of the particle from the well on the parameters of the driving force. In the absence of the driving term, a particle which is initially located at $x(0)>0$ will remain bounded in the x positive region and so it will never escape from the well. When the driving term is introduced, the dynamical evolution cannot be described in a simple analytical form. In Ref. [2] the dynamics is discussed and a sufficient condition for the particle not to reach $x=-\infty$ during any period of the external force is obtained. For large values of S ($S>\frac{1}{4}$), the particle trajectory may cross the point $x=0$ towards the negative x region, but if $x(T)-x(0)>0$ (T is the period of the external force), then the particle cannot leave the well forever. Dayan, Gittermann, and Weiss solved numerically Eq. (1) and they find different regions in the S-T parameter space where the particle either describes bounded trajectories or it leaves the well permanently after one or more external cycles.

The additional of noise implies that the particle leaves the well sooner or later regardless of the amplitude or frequency of the driving term. Dayan, Gittermann, and Weiss show how the average escape time depends on the parameters of the driving force and the strength of the white noise. The purpose of the present work is to extend the analysis of Dayan, Gittermann, and Weiss in two aspects. First we will calculate not only the average escape time but also the probability distribution of escape times for a white-noise case This is done in Sec. II. Second, in Sec. III, we will replace the white noise by an Ornstein-Uhlenbeck process in order to study the effects due to the finite correlation time of the noise. The details of the procedure followed to generate this type of noise will be presented in the Appendix.

II. ESCAPE TIMES WITH WHITE NOISE

Let us now analyze the influence of an additive white noise on the dynamics of the model described by Eq. (1). We then consider the Langevin equation

$$\frac{dx(t)}{dt} = -U'(x) + S\cos(\Omega t) + \xi(t)\ , \tag{2}$$

where $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(s) \rangle = 4\ D^2\delta(t-s)$.

To study the combined effects of noise, nonlinearity, and the external force we will resort to a numerical solution of Eq. (2). Following the ideas of Weiss and coworkers, we will consider that the particle has abandoned the well when, in the course of time, its position reaches a sufficiently large negative value. As in Ref. [2] we will take $x=-10$ to be large enough, because once this value has been reached, the probability of recrossing the position $x=0$ is negligibly small. We generate $N=5000$ stochastic trajectories by numerically solving the Langevin equation using a standard technique [3]. Then, we count the number of trajectories $N(t_i)$ crossing the point $x=-10$ for the first time in the interval t_i and t_i+h, where h is the integration step. This number is propor-

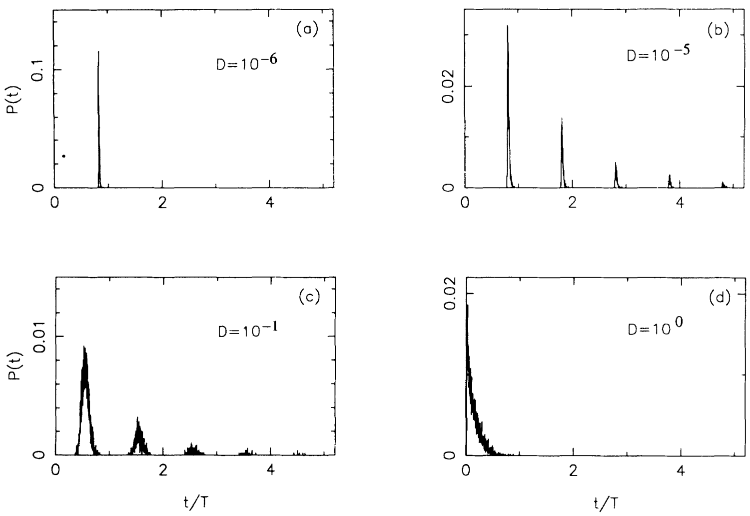

FIG. 1. Probability distribution of first-passage times for the white-noise case. Values of noise intensity range from $D=10^{-6}$ (a) to $D=10^{0}$ (d). $S=0.3$ and $\Omega=0.131\,97$ in all cases.

tional to the probability of crossing the point $x=-10$ for the first time at time t_i. The knowledge of this probability $P(t)$ allows us to compute the mean first passage time according to the expression,

$$\langle t \rangle = \sum_i \frac{t_i N(t_i)}{N} . \tag{3}$$

In Fig. 1, we plot the distribution of first passage times $P(t)$ for several values of the noise strength D and for the parameters $S=0.3$ and $\Omega=0.131\,97$. As shown in Ref. [2] these parameters correspond to a situation where the particle escapes from the well before the end of the first driving cycle in the absence of noise. When D is very small, the average escape time partially coincides with the escape time in the deterministic limit. $P(t)$ is then very sharply peaked around this value as shown in Fig. 1(a). As the noise is increased, the probability distribution changes its character to a multimodal distribution 1(b). This feature reflects that the noise perturbs the deterministic trajectory in such a way that there are trajectories which remain inside the well after the first cycle and cross the point $x=-10$ after subsequent periods of the driving force. Consequently, one should expect an increase in the average escape time. As the noise is further increased 1(c), the several peaks of the distribution become broader and their heights smaller. This reflects the competition between the diffusive effect of the noise and the confining effect of the potential well, which is periodically modulated by the driving term. For sufficiently large noise, the diffusive effect is so strong that, basically, all the trajectories escape before the end of the first period, giving rise then to a monomodal $P(t)$ 1(e). The average escape time should become much shorter than T as the probability of escape for short times is quite substantial.

The qualitative changes in the shape of $P(t)$ imply a nonmonotonic behavior of $\langle t \rangle$ with D as can be seen in Fig. 2. The results plotted there match quite well those reported by Dayan, Gittermann, and Weiss, except for very large values of D. This discrepancy may be due to the smaller number of trajectories considered by these authors. Notice that because the multipeak structure of the escape time distribution for a wide range of D, the mean escape time contains a limited amount of information about the system dynamics, in contrast with the situation of very small or very large D.

In Fig. 3 we show the results of our calculations for $S=0.3$ and $\Omega=0.131\,98$. In the noise-free limit, these parameters correspond to a situation where the particle never leaves the well permanently [2]. Therefore, one should expect that for very small D the probability for escaping within each external cycle is extremely small, although nonzero. Also, its value should be very independent of the number of cycles elapsed. These features can be observed in 3(a). As a consequence, $\langle t \rangle$ should increase tremendously as D goes to zero, as shown in Fig. 2. As the noise is increased, more and more trajectories will leave the well permanently within the first cycle, until for a large enough D, the distribution presents just one peak. Therefore, for large values of the noise, the influence of the value of the driving frequency is lost.

III. THE INFLUENCE OF AN ORNSTEIN-UHLENBECK NOISE

Let us now explore the consequences of having a noise with a finite correlation time by replacing the white noise of Eq. (2) by an Ornstein-Uhlenbeck (OU) noise $v(t)$ with $\langle v(t) \rangle = 0$ and $\langle v(t)v(s) \rangle = Q^2 \exp(-|t-s|/\tau)$. Again, we will carry out a numerical solution of the Langevin

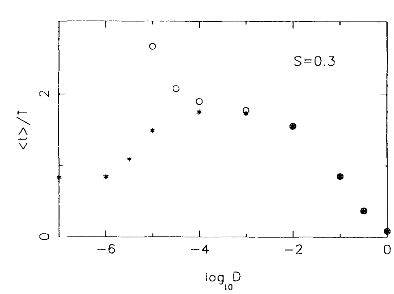

FIG. 2. Mean escape time as a function of noise intensity D for the white-noise case. ∗: $\Omega=0.131\,97$; ○: $\Omega=0.131\,98$.

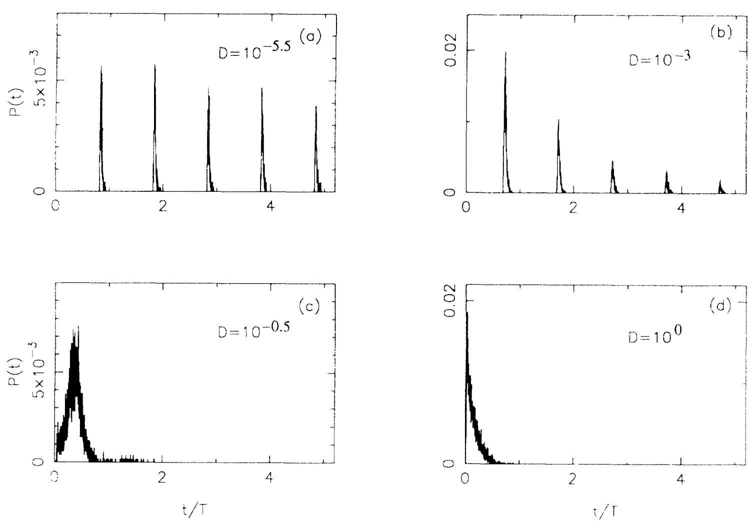

FIG. 3. Probability distribution of first-passage times for the white-noise case. $S=0.3$ and $\Omega=0.131\,98$ in all panels.

equation by generating stochastic trajectories. The details about our procedure for generating an OU noise are presented in the Appendix. We now have two noise parameters which can be independently varied. We will consider situations where the noise correlation time is either larger, equal to, or smaller than the period of the external force. The white-noise case corresponds to a noise correlation time much smaller than T.

In Fig. 4 we plot the probability distribution of the times of first passage through the position $x=-10$, for $\Omega=0.131\,97$ and $S=0.3$. The noise correlation frequency is $\tau^{-1}=0.1\ \Omega$. For very small Q, all the trajectories leave the well at the deterministic escape time. As Q is increased, $P(t)$ acquires a multipeak structure indicating that the trajectories escape in very many successive periods of the external force, as shown in 4(a) and 4(b). This feature is due to the fact that if the noise is not able to make the trajectory leave the well permanently in one cycle, it will be quite improbable that it does so in the next one due to the large correlation time of the noise compared to the period of the modulating force. By contrast, in the white-noise case all the particles leave the well after just a few cycles for the range of noise strength where $P(t)$ is multimodal. Consequently, the mean first passage time increases very quickly with the noise and reaches a maximum value, which is roughly 50% larger than the maximum value observed in the white-noise case as seen in Fig. 5. A further increase of the noise leads essentially to a broadening of the peaks [Fig. 4(c)], but there is still a non-negligible probability for the particle to escape after quite a few external cycles. $\langle t \rangle$ starts then to decrease but with higher values than in the white-noise case. For very large Q, the probability distribution is very broad, indicating that, even for large noise strengths, the probability for escaping at large times is small but nonzero [Fig. 4(d)].

These changes in the structure of $P(t)$ explain the nonmonotonic behavior of $\langle t \rangle$ with the noise strength, characteristic of what has been termed "transient stochastic resonance" by the authors of Ref. [2]. The main effect of a large noise correlation time is to enhance the height of the resonance peak of $\langle t \rangle$ with respect to that obtained in the white-noise case. We have also explored cases for noise correlation frequencies equal or larger than the driving frequency and we find that the behavior of $P(t)$ is then very similar to that obtained with white

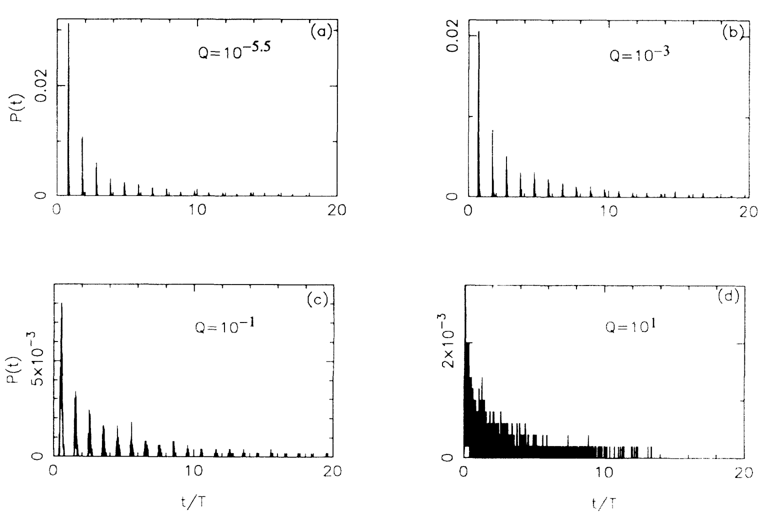

FIG. 4. Probability distribution of first-passage times for an OU noise. $S=0.3$, $\Omega=0.131\,97$, and $\tau^{-1}=0.1\ \Omega$ in all panels. The noise intensity Q ranges from $10^{-5.5}$ (a) to 10^{1} (d).

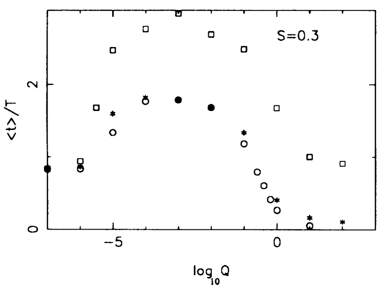

FIG. 5. Mean escape times as a function of noise intensity Q for an OU noise. $\Omega=0.131\,97$ in all panels. $*$: $\tau^{-1}=1\ \Omega$; $\circ$: $\tau^{-1}=10\ \Omega$; $\square$: $\tau^{-1}=0.1\ \Omega$.

noise. As shown in Fig. 5, the behavior of $\langle t \rangle$ for these cases is very similar to the one shown in Fig. 2 for the white-noise case.

IV. CONCLUDING REMARKS

In this work, we have analyzed the statistical distribution of escape times of a particle out of a metastable well, subject to a sinusoidal time-dependent driving term, following the ideas of Weiss and co-workers. We have considered two kinds of noise, a white noise and an OU noise with finite correlation time. In both cases, and for some values of the parameters of the driving term, the average escape time shows a maximum for some value of the noise strength. This phenomenon is associated to the change in the structure of the probability distribution of escape times. For very small noise strength, the effect of the noise on the deterministic trajectory is negligible, but as D is increased, $P(t)$ acquires a multipeak structure, with a corresponding sharp increase in $\langle t \rangle$. This effect is more pronounced for an OU noise with a correlation time much larger than the external period, due to the fact that a particle might remain confined after very many external cycles, although it should have escaped during the first cycle in the deterministic limit. The decrease of $\langle t \rangle$ with a further increase of the noise strength is due to the diffusive effect of the noise which eventually makes $P(t)$ change to a monomodal structure with its maximum located at very short times.

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APPENDIX

Let us consider a random quantity $v(t)$ obeying the Langevin equation

$$\frac{dv(t)}{dt}=-\frac{1}{\tau}v(t)+\eta(t)\ , \tag{A1}$$

where $\eta(t)$ is a white noise with zero mean and correlation function $\langle \eta(t)\eta(s)\rangle=2\ (Q^2/\tau)\delta(t-s)$. Solving Eq. (A1) for a given initial condition $v(0)$ and averaging over the realizations of the white noise one finds [4]

$$\begin{aligned}\langle v(r)\rangle &= v(0)e^{-t/\tau}\ ,\\ \langle v(t_1)v(t_2)\rangle &= e^{-(t_1+t_2)/\tau}v^2(0)\\ &\quad +Q^2[e^{-(t_1-t_2)/\tau}-e^{-(t_1+t_2)/\tau}]\ .\end{aligned} \tag{A2}$$

If the initial condition is picked from a Gaussian distribution with zero average and width Q^2 and we average over this initial distribution, we find the usual properties of an OU process, i.e.,

$$\begin{aligned}\overline{\langle v(t)\rangle} &= 0\ ,\\ \overline{\langle v(t)v(s)\rangle} &= Q^2e^{-|t-s|/\tau}\ .\end{aligned} \tag{A3}$$

We also notice that if the initial condition is set as $v(0)=0$ and we wait for $t \gg \tau$, we have that the correlation properties of the $v(t)$ process, after this transient, are the same as those of an OU process. This observation leads us to the following numerical scheme to generate an OU process. We define the quantities

$$B_n=B(t_n)+\int_{t_n}^{t_n+h}\eta(\tau)d\tau\ , \tag{A4}$$

with statistical properties

$$\begin{aligned}\langle B_n\rangle &= 0\ ,\\ \langle B_nB_m\rangle &= 2\frac{Q^2h}{\tau}\delta_{nm}\ .\end{aligned} \tag{A5}$$

A discrete version of the formal solution of (A1) with $v(0)=0$ is

$$\begin{aligned}v_n &= e^{-h/\tau}v_{n-1}+B_{n-1}\ ,\\ v_1 &= B_0\ .\end{aligned} \tag{A6}$$

This iteration scheme allows us to generate values of $\{v_n\}$ that, for a sufficiently range n, correspond to the OU process of Eq. (A3). These aged values for the v process are then used to generate stochastic trajectories of $x(t)$, by solving the Langevin equation for $x(t)$ with an OU noise term.

[1] F. Moss, in *Some Problems in Statistical Physics,* edited by G. H. Weiss (SIAM, Philadelphia, 1992); in Proceedings of the NATO Advanced Research Workshop: Stochastic Resonance in Physics and Biology, San Diego, 1992, edited by F. Moss, A. Bulsara, and M. F. Schlesinger [J. Stat. Phys. **70**, 1 (1993)].
[2] I. Dayan, M. Gittermann, and G. H. Weiss, Phys. Rev. A **46**, 757 (1992); M. Gittermann and G. H. Weiss, J. Stat. Phys. **70**, 107 (1993).
[3] H. S. Greenside and E. Helfand, Bell Syst. Tech. J. **60**, 1927 (1981); E. Helfand, *ibid.* **58**, 2289 (1979); W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes* (Cambridge University Press, Cambridge, England, 1986).
[4] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).